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BORDISM OF G-MANIFOLDS AND INTEGRALITY THEOREMS

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WE STUDY bordism of G-manifolds from a new point of view.

Our aim is to combine the geometric approach of Conner and Floyd (see [9], [10], [11], [13]) and the K-theory approach which is contained in papers by Atiyah, Bott, Segal and Singer ([1], [2], [3]). For simplicity of exposition we restrict to unitary cobordism.

We develop cobordism analogue of K-theory integrality theorems and show their relation to the results of Conner and Floyd. We get a systematic and conceptual understanding of various results about (unitary) G-manifolds.

We now describe our techniques and results. In Section 1 we define equivariant cobordism $U_G{}^*(X)$ along the lines of G. W. Whitehead [23], using all representations of the compact Lie group G for suspending. We construct a natural transformation

$$\alpha\colon U_G{}^*(X) \to U^*(EG \times_G X)$$

of multiplicative equivariant cohomology theories which preserves Thom classes. Special cases of α have been studied by Boardman [6] and Conner [9]. In particular we answer a question of Boardman ([6], p. 138).

The computation of α is interesting and very difficult in general. We have only partial results for cyclic groups. It is here that the methods of Atiyah–Segal [2] come into play: the fixed point homomorphism (Section 2) and localization (Section 3).

We consider the set $S \subset U_G{}^*$ of Euler classes of representations (considered as bundles over a point) without trivial direct summand. The first main theorem is the computation of the localization $S^{-1}U_G{}^*$ in terms of ordinary cobordism of suitable spaces.

The Pontrjagin–Thom construction gives a homomorphism

$$i\colon \mathscr{U}_*{}^G \to U_*{}^G$$

from geometric bordism $\mathscr{U}_*{}^G$ of unitary G-manifolds to homotopical bordism. The map i is by no means an isomorphism (due to the lack of usual transversality theorems). The elements $x \in S$, $x \neq 1$, do not lie in the image of i. One might conjecture that $U_*{}^G$ is generated as an algebra by S and the image of i. We prove this for cyclic groups $\mathbf{Z}_p$ of prime order p (Section

5). We also compute $U_G{}^*$ for these groups by embedding it into an exact sequence ($G = \mathbf{Z}_p$)

$$o \to U^* \to U_G{}^* \xrightarrow{\lambda} S^{-1}U_G{}^* \to \tilde{U}_*(BG) \to o,$$

which is analogous to exact sequences of Conner and Floyd [13]. The resulting isomorphism

$$\tilde{U}_*(BG) \cong \text{Cokernel } \lambda$$

gives a very convenient description of the relations in $\tilde{U}_*(BG)$ (compare [10]). It can be used to prove that invariants of type v of Atiyah–Singer [3, p. 587], characterize unitary bordism of G-manifolds (G cyclic). The product structure which $\tilde{U}_{2n-1}(BG)$ inherits from the above isomorphism has been found by Conner [9, pp. 80–81].

Not every bundle can appear as normal bundle to the fixed point set of a G-manifold. The bundle has to satisfy various "integrality relations" which are derived from our localization theorem. We prove that for $G = \mathbf{Z}_p$ a bundle appears (up to bordism) as normal bundle to the fixed point set if and only if it satisfies these integrality relations. We list some theorems of Conner–Floyd [11] which are easily accessible through our techniques: 27.1, §30, §31, 43.6, §46. See also [16].

Finally we use results of Conner [9] and Stong [22], Hattori [19] to show: K-theory characteristic numbers characterize unitary bordism of involutions.

The intention of the present paper is to describe some general ideas. Various applications will appear elsewhere.

I am grateful to the referee who read my manuscript very carefully and made a number of suggestions which lead to an improvement of the presentation especially of Section 5.

§1. EQUIVARIANT COBORDISM

We sketch the beginnings of equivariant unitary cobordism. For a detailed description see [17].

Let G be a compact Lie group. Let $D(G)$ be the set of representations of G in some standard vector space $\mathbf{C}^n, n = o, 1, 2, \ldots$. We define a pre-order on $D(G)$ as follows: $V < W$ if and only if V is isomorphic to some G-submodule of W. We list without proof the following simple lemma.

LEMMA 1.1. *Any two isomorphisms $f, g\colon V \to W$ of complex G-modules are homotopic as G-isomorphisms.*

Let V be a complex G-module. We denote by $V^c = V \cup \{\infty\}$ its one-point compactification which we consider as pointed G-space with base point ∞. We write $|V|$ for $\dim_{\mathbf{C}} V$. Let $V, W \in D(G)$ and suppose $V < W$. So there is a $U \in D(G)$ with $U \oplus V \cong W$. Let $\gamma_k\colon E_k(G) \to B_k(G)$ be the universal k-dimensional complex G-vector bundle ([14]) and $M_k(G)$ its Thom-space considered as pointed G-space. Let U also denote the G-bundle $U \times X \to X$ for any G-space X. A classifying map

$$f_U : U \oplus \gamma_k \to \gamma_{|U|+k}$$

induces a pointed G-map

$$g_U = M(f_U)\colon U^c \wedge M_k(G) = M(U \oplus \gamma_k) \to M(\gamma_{|U|+k}) = M_{|U|+k}(G),$$

where in general $M(\xi)$ denotes the Thom-space of the bundle ξ.

We define a natural transformation $b^n_{W,V}$ by

$$\begin{aligned} b^n_{W,V}(X, Y)\colon & [V^c \wedge X, M_{|V|+n}(G) \wedge Y]^o_G \xrightarrow{(1)} \\ & [U^c \wedge V^c \wedge X, U^c \wedge M_{|V|+n}(G) \wedge Y]^o_G \xrightarrow{(2)} \\ & [W^c \wedge X, U^c \wedge M_{|V|+n}(G) \wedge Y]^o_G \xrightarrow{(3)} \\ & [W^c \wedge X, M_{|W|+n}(G) \wedge Y]^o_G . \end{aligned}$$

Here $[-,-]^o_G$ denotes pointed G-homotopy set. X and Y are pointed G-spaces. (1) is smash-product with U^c. The G-homeomorphisms $U^c \wedge V^c \cong (U \oplus V)^c \cong W^c$ induce (2). The map g_U induces (3). Because of Lemma 1.1 $b^n_{W,V}$ does not depend on the choice of the isomorphism $U \oplus V \cong W$. If $U < V < W$ then

$$b^n_{W,V} \circ b^n_{V,U} = b^n_{W,U} .$$

Therefore the transformations $b^n_{V,U}$ form a direct system over $D(G)$. We call the direct limit

$$\tilde{U}_G{}^{2n}(X; Y).$$

If SX is the suspension with trivial G-action on the suspension coordinate we put

$$\tilde{U}_G{}^{2n-1}(X; Y) = \tilde{U}_G{}^{2n}(SX; Y).$$

We make the usual conventions: If S^o is the zero-sphere we put $\tilde{U}_G{}^k(X) = \tilde{U}_G{}^k(X; S^o)$, $\tilde{U}_k{}^G(Y) = \tilde{U}_G{}^{-k}(S^o; Y)$, and $U_G{}^k(Z) = \tilde{U}_G{}^k(Z^+)$, where Z^+ is Z with a separate base point, $U_G{}^k = U_G{}^k(\text{Point})$, and $U_k{}^G(X, Y) = \tilde{U}_k{}^G(X/Y)$ if $Y \subset X$ is a G-cofibration.

The $\tilde{U}_G{}^k(-; Y)$ for fixed Y form an equivariant cohomology theory (Bredon [4]). There are pairings

$$\tilde{U}_G{}^r(X; Y) \otimes \tilde{U}_G{}^s(X'; Y') \to \tilde{U}_G{}^{r+s}(X \wedge X'; Y \wedge Y').$$

All this is well known when there is no group G (Whitehead [23]) and quite analogously here. The $\tilde{U}_G{}^k(-)$ form a multiplicative cohomology theory.

If ξ is a complex n-dimensional G-vector bundle over X the classifying map of ξ induces a map $M(\xi) \to M_n(G)$ which represents the Thom class

$$t(\xi) \in \tilde{U}_G{}^{2n}(M(\xi))$$

of ξ. If $s\colon X^+ \to M(\xi)$ is the zero section of ξ we call $e(\xi) = s^* t(\xi)$ the Euler class of ξ. If V is a complex G-module we can view V as a bundle over a point and so we have a Thom class $t(V) \in \tilde{U}_G{}^{2|V|}(V^c)$ and an Euler class $e(V) \in U_G{}^{2|V|}$. Multiplication with $t(V)$ gives a suspension isomorphism

$$\sigma(V)\colon \tilde{U}_G{}^k(X) \cong \tilde{U}_G{}^{k+2|V|}(V^c \wedge X)$$

for any $V \in D(G)$.

Now we introduce an important natural transformation. Let EG be a free contractible G-space such that $EG \to EG/G$ is numerable (Dold [18], p. 226). We only consider left G-actions.

PROPOSITION 1.2. *There exists a canonical natural transformation of equivariant cohomology theories*

$$\alpha\colon \tilde{U}_G{}^*(Z) \to \tilde{U}^*((Z \wedge EG^+)/G).$$

α preserves multiplication and Thom classes. If Y is a compact free G-space and $Z = Y^+$ then $\alpha(Z)$ is an isomorphism.

($\tilde{U}^*(K)$ is the usual unitary cobordism ring of the pointed space K. If $Z = M(\xi)$ then $(Z \wedge EG^+)/G$ can be considered as the Thom space of the bundle $(\xi \times 1_{EG})/G$.)

Proof. The classifying map of the $U(k)$-bundle

$$(E_k(G) \times EG)/G \to (B_k(G) \times EG)/G$$

induces a map of the corresponding Thom spaces

$$r_k\colon (M_k(G) \wedge EG^+)/G \to M_k,$$

where $M_k = M_k(\{e\})$. We use r_k to construct the natural transformation

$$\begin{aligned} &[V^c \wedge Z, M_{k+|V|}(G)]_G^o \longrightarrow \\ &[(V^c \wedge Z \wedge EG^+)/G, (M_{k+|V|}(G) \wedge EG^+)/G]_G^o \xrightarrow{\ r_{k+|V|}\ } \\ &[(V^c \wedge Z \wedge EG^+)/G, M_{k+|V|}]^o. \end{aligned}$$

By definition of the cobordism groups the last homotopy set maps naturally into

$$\tilde{U}^{2k+2|V|}((V^c \wedge Z \wedge EG^+)/G).$$

Using a canonical relative Thom isomorphism the last group is isomorphic to

$$\tilde{U}^{2k}((Z \wedge EG^+)/G).$$

Hence we got maps

$$[V^c \wedge Z, M_{k+|V|}(G)]_G^o \to \tilde{U}^{2k}((Z \wedge EG^+)/G)$$

which yield the desired map α if we pass to the direct limit. It is clear that α is natural, multiplicative and preserves Thom classes. The assertion about $\alpha(Y^+)$ follows by applying the results of [15].

Remark. More generally we could have constructed a natural transformation

$$\alpha\colon \tilde{U}_G{}^*(X; Y) \to \tilde{U}^*((X \wedge EG^+)/G; (Y \wedge EG^+)/G).$$

We call transformations of this type *bundling transformations*. Similar maps α exist for other cobordism theories, e.g. "unoriented" cobordism. For $G = Z_2$ and $Z = S^o$ essentially this map was studied by Boardman [5], [6]. See also Conner [9, 12]. Our approach gives an immediate insight into the multiplicative property (see the question of Boardman, [6 p. 138]).

Let Y be a G-space and $\mathcal{U}_n{}^G(Y)$ the bordism group of n-dimensional unitary singular G-manifolds in Y. In the manner of Conner and Floyd [11] one constructs an equivariant homology theory.

There is a natural transformation of equivariant homology theories

$$i\colon \mathcal{U}_*{}^G(-) \to U_*{}^G(-),$$

defined as in Conner–Floyd [11], 12. It is sufficient to indicate the construction of i for the coefficients of the theory. Given a unitary G-manifold M of dimension n. If n is even there exists a G-embedding $M \subset V$ of M in some complex G-module $V \in D(G)$ such that the normal bundle ν has the correct complex structure. A classifying map for ν gives in the usual way (Pontrjagin–Thom construction) a map

$$V^c \to M(\nu) \to M_r(G),$$

$r = |V| - \frac{1}{2}\dim M$. This map shall represent $i[M]$. If n is odd we embed into $V \oplus \mathbf{R}$, $V \in D(G)$.

Remark. The map i is not an isomorphism, if G is non-trivial (compare Theorem 3.1).

PROPOSITION 1.3. *Let Y be a free G-space. Then $i\colon \mathcal{U}_n{}^G(Y) \to U_n{}^G(Y)$ is an isomorphism.*

Proof. By standard approximation techniques it is enough to consider the case that Y is a G-manifold. The group $U_n{}^G(Y)$ is the direct limit over homotopy sets of the form

$$[V^c, M_k(G) \wedge Y^+]_G^o = [V^c, M(\gamma_k \times \mathrm{id}(Y))]_G^o.$$

But γ_k may be approximated by G-bundles over differentiable manifolds (e.g. Grassmannians) and hence $M(\gamma_k \times \mathrm{id}(Y))$ by Thom spaces which are in a neighbourhood of the zero-section free G-manifolds. But for G-maps between free G-manifolds usual transversality arguments apply, and we can immitate Thom's proof that geometric bordism may be described by homotopy groups of Thom spaces.

§2. THE FIXED POINT HOMOMORPHISM

Restriction to the fixed point set is a functor from G-spaces to spaces, compatible with homotopy in both categories. We analyse this process in our set up.

We consider the classifying space BU as a space with base point 1. Whitney-sum of vector bundles induces an H-space structure $s\colon BU \times BU \to BU$. We can assume $s(1, b) = s(b, 1) = b$ for all $b \in BU$. Let $J(G)$ be the set of isomorphism classes of non-trivial irreducible G-modules and let

$$B \subset \prod_{j \in J(G)} BU$$

be the subspace of the product consisting of points which have only finitely many components different from the base point. Then s induces an H-space structure on B, again denoted by s and defined by

$$s((b_j), (c_j)) = (s(b_j, c_j)).$$

Let X be a compact pointed space with trivial G-action and Y a pointed G-space with fixed point set F. We use s to give $\tilde{U}^*(X; B^+)$ the structure of a U^*-algebra (cup product and Pontrjagin multiplication) and $\tilde{U}^*(X; B^+ \wedge F)$ the structure of a $\tilde{U}^*(X; B^+)$-module.

Let $R_1(G)$ be the additive subgroup of the representation ring $R(G)$ of G (Segal [21], p. 113) which is additively generated by the non-trivial irreducible representations. Let $A(G)$ be the group ring over the integers $\mathbf{Z}$ of the group $R_1(G)$. We define a grading on $A(G)$ by assigning to elements of $R_1(G)$ as degree their (virtual) dimension over the reals. Let

$$\tilde{L}_G{}^*(X; F) = \tilde{U}^*(X; B^+ \wedge F) \otimes A(G)$$

be the graded tensor product over the integers.

Our aim is to describe a homomorphism

$$\varphi\colon \tilde{U}_G{}^*(X; Y) \to \tilde{L}_G{}^*(X; F)$$

induced by "restriction to the fixed point set." We need the next lemma. We use the following notation: Let $V(G)$ be the set of isomorphism classes of complex G-modules. If $V \in V(G)$ let V_o be the trivial and V_1 be the non-trivial direct summand of V. Let $Z(V)$ be the automorphism group of the G-module V.

LEMMA 2.1. *The fixed point set of the Thom space $M_n(G)$ is homotopy equivalent to*

$$\bigvee (MU(|V_o|) \wedge BZ(V_1)^+).$$

The sum $\bigvee$ (in the category of pointed spaces) is taken over all $V \in V(G)$ with $|V| = n$.

Proof. Let γ_n over $B_n(G)$ be the universal complex n-dimensional G-vector bundle. The universal property of γ_n implies the following facts. The path-components of $B_n(G)^G$ are classifying spaces $BZ(V)$, $|V| = n$. The restriction of γ_n to $BZ(V)$ is isomorphic to a bundle of the form

$$\gamma(o) \times \gamma(1)\colon E(o) \times E(1) \to BZ(V_o) \times BZ(V_1) = BZ(V).$$

The bundle $\gamma(o)$ is the usual $|V_o|$-dimensional universal vector bundle and $E(1)$ has only the zero section as fixed point set. As usual we put $M(\gamma_o) = MU(|V_o|)$.

Now consider the following composition of mappings which we explain in a moment

$$\begin{aligned}
&[W^c \wedge X, M_{n+|W|}(G) \wedge Y]_G^o \xrightarrow{(1)} \\
&[W_o{}^c \wedge X, (\bigvee MU(|V_o|) \wedge BZ(V_1)^+) \wedge F]^o \xrightarrow{(2)} \\
&[W_o{}^c \wedge X, (\Pi MU(|V_o|) \wedge BZ(V_1)^+) \wedge F]^o \xrightarrow{(3)} \\
&\oplus \tilde{U}^{2(|V_o|-|W_o|)}(X; BZ(V_1)^+ \wedge F).
\end{aligned}$$

Explanation. (1) is restriction to the fixed point set. We have used Lemma 2.1. The $V = V_o \oplus V_1$ run through $V \in V(G)$ with $|V| = n + |W|$. Inclusion of the sum into the product induces (2). The definition of $\tilde{U}^*(-;-)$ as a direct limit of homotopy sets gives (3).

The space $BZ(V_1)$ is homotopy equivalent to a certain product $\Pi BU(m_j), j \in J(G)$. We have a canonical map (unique up to homotopy) $BZ(V_1) \to B$ (let m_j go to infinity). If we use this map in the composition above we get a map

$$\varphi_W'\colon [W^c \wedge X, M_{n+|W|}(G) \wedge Y]_G^o \to \oplus \tilde{U}^{2(|V_o|-|W_o|)}(X; B^+ \wedge F), \qquad |V| = n + |W|.$$

We denote the V-component of $\varphi_W'(x)$ by $x(V)$ and define

$$\varphi_W\colon [W^c \wedge X, M_{n+|W|}(G) \wedge Y]_G^o \to \tilde{L}_G{}^{2n}(X; F)$$

by

$$\varphi_W(x) = \Sigma x(V) \otimes (V_1 - W_1), \qquad |V| = n + |W|.$$

One verifies that the φ_W are compatible with the limiting process and hence yield a map

$$\varphi\colon \tilde{U}_G{}^{2n}(X; Y) \to \tilde{L}_G{}^{2n}(X; F).$$

In odd dimensions we replace X by SX and proceed as above.

LEMMA 2.2. *The map φ is a homomorphism of $\tilde{U}^*(X)$-modules of degree zero. If $Y = S^o$ is the pointed zero sphere then φ is a homomorphism of $\tilde{U}^*(X)$-algebras. The image of the Euler class $e(V_1)$ of V_1 under φ is $1 \otimes V_1$.*

Proof. Straightforward verification. Note that the product in $U_G{}^*$-theory comes from a pairing of Thom spaces $M_n(G) \wedge M_m(G) \to M_{n+m}(G)$. When we restrict to the fixed point set this is related to the H-space structure s on B.

§3. LOCALIZATION

Let $S \subset U_G{}^*$ be the multiplicatively closed subset which contains 1 and the Euler classes $e(V_1)$, $V \in V(G)$. According to Lemma 2.2 $\varphi(S)$ consists of invertible elements. Therefore we introduce the elements of S as denominators into $\tilde{U}_G{}^*(X; Y)$ and denote the resulting graded module of quotients by $S^{-1}\tilde{U}_G{}^*(X; Y)$ (see Bourbaki [8] for notion and notation). The universal property of the canonical map

$$\lambda\colon \tilde{U}_G{}^*(X; Y) \to S^{-1}\tilde{U}_G{}^*(X; Y)$$

provides us with a unique homomorphism

$$\Phi\colon S^{-1}\tilde{U}_G{}^*(X; Y) \to \tilde{L}_G{}^*(X; F)$$

with the property $\Phi\lambda = \varphi$. (Here X, Y, and F have the same meaning as in Section 2.)

THEOREM 3.1. *Φ is an isomorphism.*

Proof. We construct an inverse Ψ to Φ. Given $z \in \tilde{U}^t(X; B^+ \wedge F)$. Assume for the moment that t is even, $t = 2n$. The element z is represented by a map

$$f\colon S^{2r} \wedge X \to MU(n + r) \wedge B^+ \wedge F.$$

But X is compact, hence there exists a $V \in V(G)$ with $|V_o| = n + r$ and such that f factorises up to homotopy over

$$MU(|V_o|) \wedge BZ(V_1)^+ \wedge F. \tag{1}$$

We denote the induced map of $S^{2r} \wedge X$ into the space (1) again by f. The space (1) has an inclusion f_1 into the fixed point set of $M_p(G) \wedge Y$, $p = |V|$, according to Lemma 2.1. We regard $f_1 f$ as a G-map

$$S^{2r} \wedge X \to M_p(G) \wedge Y,$$

representing an element

$$[f_1 f] \in \tilde{U}_G{}^q(X; Y), \qquad q = 2|V| - 2r.$$

One can see that the element

$$\lambda[f_1 f] \cdot e(V_1)^{-1} \in (S^{-1}\tilde{U}_G{}^*(X;\, Y))^{2n}$$

depends only on z and not on the choice of f and V. We define an $A(G)$-linear map Ψ by

$$\Psi(z \otimes 1) = \lambda[f_1 f] \cdot e(V_1)^{-1}.$$

(If t is odd, replace X by SX.)

The construction of φ immediately gives $\varphi[f_1 f] = z \otimes V_1$ and therefore

$$\begin{aligned}\Phi\Psi(z \otimes 1) &= \Phi(\lambda[f_1 f] \cdot e(V_1)^{-1}) \\ &= \varphi[f_1 f] \cdot \Phi e(V_1)^{-1} \\ &= (z \otimes V_1)(1 \otimes (-V_1)) \\ &= z \otimes 1.\end{aligned}$$

To prove $\Psi\Phi = \mathrm{id}$ it is sufficient to prove $\Psi\Phi\lambda = \lambda$, i.e. $\Psi\varphi = \lambda$. We start with $x \in \tilde{U}_G{}^*(X;\, Y)$ represented by

$$f\colon W^c \wedge X \to M_{n+|W|}(G) \wedge Y.$$

Suppose we have

$$\varphi x = \Sigma x(V) \otimes (V_1 - W_1)$$

as in the definition of φ. By definition of Ψ the element $\Psi(\Sigma x(V) \otimes V_1)$ is given by $\lambda[f']$, where f' is the map fi, $i\colon W_o{}^c \wedge X \subset W^c \wedge X$. On the other hand f' represents the image of x under

$$\tilde{U}_G{}^*(X;\, Y) \xrightarrow[\sigma(W)]{} \tilde{U}_G{}^*(W^c \wedge X;\, Y) \xrightarrow[i^*]{} \tilde{U}_G{}^*(W_o{}^c \wedge X;\, Y) \xrightarrow[\sigma(W_o)^{-1}]{} \tilde{U}_G{}^*(X;\, Y).$$

But this composition obviously is multiplication with the Euler class $e(W_1)$. Put together we have

$$\Psi\varphi x = \lambda[f']e(W_1)^{-1} = \lambda x \cdot e(W_1) \cdot (W_1)^{-1} = \lambda x.$$

COROLLARY 3.2. *The elements of S are different from zero. $S^{-1}U_G{}^*$ is a free U_*-module.*

We go on to give a more geometric interpretation of Theorem 3.1. If $X = Y = S^o$ we have an isomorphism

$$S^{-1}U_G{}^* \cong U_*(B) \otimes A(G).$$

We give another description of elements in the right hand group. Let M be a compact unitary manifold without boundary and with trivial G-action. Let $\alpha \in K_G(M)$ (equivariant K-theory of M, see Segal [20]) be an element without trivial summand: We can write α in the form $\alpha = E - F$, where E is a complex G-vector bundle over M and F is a trivial G-vector bundle of the form $pr\colon M \times V \to M$, with V a G-module. Moreover we can assume that E and F do not have direct summands with trivial G-action. Put

$$E \cong \bigoplus_{W \in J(G)} (E_W \otimes W)$$

(Segal [20, Proposition 2.2]) and let

$$f_W : M \to BU(m_W), \qquad m_W = \dim E_W,$$

be a classifying map for E_W. Then

$$f : M \xrightarrow[(f_W)]{} \Pi BU(m_W) \to B$$

represents a bordism element $x \in U_*(B)$. Let M be connected and E_m, F_m be the fibre of E, F over $m \in M$ considered as G-modules. We asign to the pair (M, α) the element

$$\Gamma(M, \alpha) := x \otimes (E_m - F_m) \in U_*(B) \otimes A(G).$$

It is clear that any $y \in U_*(B) \otimes A(G)$ is a sum of elements of the form $\Gamma(M, \alpha)$. Hence we can view $U_*(B) \otimes A(G)$ as a suitable bordism group of pairs (M, α).

Let $q: M \to P$ be the projection onto a point. Since unitary manifolds are orientable with respect to the cohomology theory $U^*(-)$ we have a Gysin homomorphism

$$q_! : U_G{}^*(M) \to U_G{}^*$$

of degree $-\dim M$.

THEOREM 3.3. *We have* $\Psi\Gamma(M, \alpha) = e(F_m)^{-1} q_!(e(E))$, *where* $e(E) \in U_G{}^*(M)$ *is the Euler class of* E.

We omit the simple proof and list only an easy consequence. If we are given a natural transformation of multiplicative equivariant cohomology theories

$$\alpha: U_G{}^*(-) \to h_G{}^*(-)$$

which maps Thom classes to Thom classes, then α is also compatible with Gysin homomorphisms and Theorem 3.3 gives a method for computing the localized map $S^{-1}\alpha$. The two most important examples of such transformations are the bundling transformation

$$\alpha: U_G{}^* \to U^*(BG)$$

of Section 1 and the equivariant analogue

$$\mu: U_G{}^* \to K_G{}^*$$

of the Conner–Floyd map ([12], Ch. I.5).

§4. INTEGRALITY

The localization Theorem 3.1 is intimately connected with the Conner–Floyd approach to equivariant bordism. Geometrically the restriction to the fixed point set defines a homomorphism

$$\varphi_1 : \mathscr{U}_n{}^G \to \oplus\, U_{2t}(\Pi BU(t_V))$$

where the sum is taken over all t, t_V with $n = 2t + 2\Sigma t_V |V|$, $V \in J(G)$. We recall its definition (see also Conner–Floyd [13, 5.]).

Let M be a unitary G-manifold and let F denote a component of the fixed point set. The normal bundle to F in M has a canonical G-invariant complex structure, hence has the form $\oplus(V \otimes N_V)$, $V \in J(G)$. Let $f_V : F \to BU(t_V)$ be a classifying map for N_V. Then $\varphi_1[M]$ is defined to be the sum over all F of the singular manifolds

$$(f_V): F \to \Pi BU(t_V).$$

We have an inclusion

$$w: \oplus U_{2t}(\Pi BU(t_V)) \to U_*(B) \otimes A(G)$$

mapping the element $y \in U_{2t}(\Pi BU(t_V))$ to $b(y) \otimes (-\Sigma t_V V)$, where

$$b\colon U_{2t}(\Pi BU(t_V)) \to U_{2t}(B)$$

is the canonical map. Let $\nu\colon BU \to BU$ denote the "inverse" of the H-space BU (with $\nu 1 = 1$) and $n\colon B \to B$ the map induced by $\Pi_j \nu\colon \Pi_j BU \to \Pi_j BU$.

PROPOSITION 4.1. *The following diagram is commutative.*

$$\begin{tikzcd} \mathcal{U}_n^G \arrow[r, "i"] \arrow[d, "w\varphi_1"'] & U_n^G \arrow[d, "\varphi"] \\ U_*(B) \otimes A(G) \arrow[r, "U_*(n)\otimes \mathrm{id}"'] & U_*(B) \otimes A(G) \end{tikzcd}$$

Proof. Given a G-manifold M of even dimension, the definition of i requires an embedding $M \subset V$, where V is a complex G-module. The image $i[M]$ is represented by a map $h\colon V^c \to M_m(G)$ which is transverse to the zero section and such that the restriction of h to M is a classifying map of the normal bundle $\nu_{M,V}$ of M in V. If we restrict h to the fixed point set we get a map (with $W = V_o$)

$$h_1\colon W^c \to \bigvee_{(m)} MU(m_o) \wedge (\Pi_j BU(m_j))^+$$

which is transverse to the sum C of the $BU(m_o) \times \Pi_j BU(m_j) = :B_{(m)}$. (Here (m) runs through (m_o, m_j) with $m_o + \Sigma |V_j| m_j = m, j \in J(G)$.) Moreover $h_1^{-1}C$ is the fixed point set F of M. The map h_1 induces $F \to C$ which is a classifying map for $\nu_{M,V} | F$ and which decomposes into a sum of $F_{(m)} \to B_{(m)}$. We have the equality of bundles

$$(1) \qquad \nu_{F,W} | F_{(m)} \oplus \nu_{W,V} | F_{(m)} \cong \nu_{F,M} | F_{(m)} \oplus \nu_{M,V} | F_{(m)}.$$

But these are bundles over a trivial G-space. Hence we have decompositions of the form

$$\nu_{F,M} | F_{(m)} = \oplus_j (V_j \otimes N_{j,(m)})$$
$$\nu_{M,V} | F_{(m)} = \oplus_j (V_j \otimes D_{j,(m)}) \oplus D_{o,(m)}$$

with trivial G-action on $D_{o,(m)}$. The equality (1) yields the following stable equivalences

$$(2) \qquad \begin{aligned} N_{j,(m)}^{-1} &\sim D_{j,(m)} \\ \nu_{F,W} | F_{(m)} &\sim D_{o,(m)} \end{aligned}$$

(N^{-1} means a bundle inverse to N). If $p_{j,(m)}$ is a stable classifying map of $D_{j,(m)}$, then $\varphi i[M]$ is

$$\Sigma_{(m)}[(p_{j,(m)} | j \in J(G))\colon F_{(m)} \to B] \otimes (\Sigma_j (m_j - k_j) V_j)$$

if we have $V = V_o \oplus \Sigma_j k_j V_j$. (Note: In our earlier notation $V_1 = \Sigma_j k_j V_j$.) If $q_{j,(m)}$ denotes a stable classifying map of $N_{j,(m)}$, then $w\varphi_1[M]$ is

$$\Sigma_{(m)}[(q_{j,(m)} | j \in J(G))\colon F_{(m)} \to B] \otimes (-\Sigma_j l_{j,(m)} V_j)$$

with $l_{j,(m)} = \dim_C N_{j,(m)}$.

From (2) we get

$$\nu q_{j,(m)} \quad \text{homotopic} \quad p_{j,(m)}$$

and from (1) we get

$$k_j = l_{j,(m)} + m_j$$

and hence commutativity of the diagram. If n is odd we embed M into $V \oplus \mathbf{R}$ and proceed as above.

Since the bundling transformation α preserves Thom classes and hence Euler classes we have an induced map $S^{-1}\alpha$. Proposition 4.1 and Theorem 3.1 have as corollary the

PROPOSITION 4.2. *If $x \in \mathcal{U}_n{}^G$ is represented by a G-manifold without stationary points, then $\alpha i x$ is in the kernel of the canonical map Λ: $U^*(BG) \to S^{-1}U^*(BG)$ (i.e. $\alpha i x$ is annihilated by some product of Euler classes contained in S).*

The contrapositive of Proposition 4.2 is a general existence theorem for fixed points on G-manifolds. If S does not contain zero divisors (e.g. G a torus) and $[M] \in \mathcal{U}_n{}^G$ is represented by a manifold without fixed points, then $\alpha i[M]=0$. In particular M bounds if we forget the G-action (compare Bott [7]).

An element $y \in U_*(B) \otimes A(G)$ is in the image of φ only if $S^{-1}\alpha(y)$ is "integral" (i.e. contained in the image of Λ: $U^*(BG) \to S^{-1}U^*(BG)$). This "integrality condition" is analogous to K-theory integrality conditions (Atiyah–Segal [2]). We say that the "integrality theorem" holds if the integrality of $S^{-1}\alpha(y)$ implies $y \in$ image φ.

§5. CYCLIC GROUPS

THEOREM 5.1. *Let G be the cyclic group $\mathbf{Z}_p$ of prime order p. Then we have:*

(a) *There exists a canonical exact sequence*

$$o \to U_n \xrightarrow[\delta]{} U_n{}^G \xrightarrow[\lambda]{} (S^{-1}U_G{}^*)_n \xrightarrow[\beta]{} \tilde{U}_{n-1}(BZ_p) \to o.$$

(b) $S^{-1}\alpha$ *induces an isomorphism*

$$\text{Cokernel } \lambda \cong \text{Cokernel } \Lambda$$

(i.e. *the integrality theorem holds*).

(c) $U_G{}^*$ *is generated (as an algebra) by the image of* i: $\mathcal{U}_G{}^* \to U_G{}^*$ *and S. The map i is injective.*

Proof. (a) Let $V_1(G)$ be the set of isomorphism classes of complex G-modules without trivial direct summand. For $V \in V_1(G)$ let $S(V)$ be the unit sphere in a G-invariant hermitian metric (we do not distinguish between elements of $V_1(G)$ and representing G-modules). We have a Gysin sequence $\Sigma(V)$

$$\cdots \to U_n{}^G \xrightarrow[e(V)\cdot]{} U^G_{n-2|V|} \to U^G_{n-1}(SV) \to U^G_{n-1} \to \cdots.$$

Here $e(V)\cdot$ means multiplication with $e(V)$.

If $W = U \oplus V \in V_1(G)$ we have a morphism $\Sigma(V) \to \Sigma(W)$ consisting of the three pieces id: $U_n{}^G \to U_n{}^G$ and $e(U)\cdot$: $U^G_{n-2|V|} \to U^G_{n-2|W|}$ and j_* : $U^G_{n-1}(SV) \to U^G_{n-1}(SW)$ with j: $SV \to SW$ the inclusion. The direct limit over these morphisms yields an exact sequence

1 $$\cdots \to U_n{}^G \xrightarrow{\lambda} (S^{-1}U_*{}^G)_n \xrightarrow{\beta} U_{n-1}(BG) \xrightarrow{\delta} \cdots$$

as follows: The limit over id: $U_n^G \to U_n^G$ is clearly U_n^G. The limit over the multiplications $e(U)\cdot$ is well known to be isomorphic to $S^{-1}U_*^G$, the isomorphism being induced by mapping $x \in U^G_{n-2|V|}$ to $e(V)^{-1}x \in (S^{-1}U_*^G)_n$. The sphere SV is a free G-space because $G = \mathbf{Z}_p$ and V has no G-trivial direct summands. We have natural isomorphisms

$$U_n^G(SV) \cong \mathscr{U}_n^G(SV) \cong U_n(SV/G)$$

(see Proposition 1.3) and the direct limit over the $U_n(SV/G)$ is $U_n(BG)$.

We use (1) to prove (a). If n is even, then $U_n(BG) = U_n$ and

$$\delta\colon U_n = U_n(BG) \to U_n^G$$

composed with the map $\varepsilon\colon U_n^G \to U_n$ which forgets the group action is multiplication by p. Hence

$$o \to U_n \to U_n^G \to (S^{-1}U_*^G)_n \to U_{n-1}(BG)$$

is exact for even n, by (1) and because U_* is torsion free. Moreover $U_o^G \to U^G_{2|V|-1}(SV) \to U_{2|V|-1}(BG)$ is seen to map 1 to the bordism class of the inclusion $SV/G \to BG$. But $\tilde{U}_*(BG)$ is generated (as U_*-module) by such elements (Conner–Floyd [10]). So we conclude that β is onto for n even. If k is odd we know by Theorem 3.1 that $(S^{-1}U_G{}^*)_k = o$, and (1) together with (a) for n even gives $U_k^G = o$. This proves (a).

(b) To prove (b) we use the cohomology form of (1) and the bundling transformation α. We have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & U_G^n \arrow[r, "e(V)\cdot"] \arrow[d, "\alpha"] & U_G^{n+2|V|} \arrow[r] \arrow[d, "\alpha"] & U_G^{n+2|V|}(SV) \arrow[r] \arrow[d, "\alpha"] & \cdots \\
\cdots \arrow[r] & U^n(BG) \arrow[r, "e(V)\cdot"] & U^{n+2|V|}(BG) \arrow[r] & U^{n+2|V|}(EG \times_G SV) \arrow[r] & \cdots
\end{tikzcd}$$

with exact rows (Gysin sequences). The right hand α is an isomorphism by Proposition 1.2. We pass to the direct limit and get (b).

(c) We have the natural transformation i relating geometrical with homotopical bordism. If we take the direct limit over the $V \in V_1(G)$ of the commutative diagrams

$$\begin{tikzcd}
\cdots \arrow[r] & \mathscr{U}_n^G(DV) \arrow[r] \arrow[d, "i"] & \mathscr{U}_n^G(DV, SV) \arrow[r] \arrow[d, "i"] & \mathscr{U}^G_{n-1}(SV) \arrow[r] \arrow[d, "i"] & \cdots \\
\cdots \arrow[r] & U_n^G(DV) \arrow[r] & U_n^G(DV, SV) \arrow[r] & U^G_{n-1}(SV) & \cdots
\end{tikzcd}$$

we get a commutative diagram

$$\begin{tikzcd}
o \arrow[r] & U_n \arrow[r] \arrow[d, "\mathrm{id}"] & \mathscr{U}_n^G \arrow[r, "\lambda'"] \arrow[d, "i"] & F_n \arrow[r, "\beta'"] \arrow[d, "\iota"] & \tilde{U}_{n-1}(BG) \arrow[r] \arrow[d, "\mathrm{id}"] & o \\
o \arrow[r] & U_n \arrow[r] & U_n^G \arrow[r, "\lambda"] & (S^{-1}U_*^G)_n \arrow[r, "\beta"] & \tilde{U}_{n-1}(BG) \arrow[r] & o.
\end{tikzcd}$$

The top sequence is Conner–Floyd's sequence relating free and arbitrary G-bordism ([13]).

We can identify F_n with

$$\oplus U_k\left(\prod_{j \in J(G)} BU(k_j)\right)$$

where the sum is taken over k, k_j with $k + 2\Sigma k_j = n$. Then λ' is taking the normal bundle to the fixed point set. The map t is the map $w\varphi_1$ of Proposition 4.1. It is injective, hence i is injective. It is obvious from Theorem 3.1 that $S^{-1}U_*^G$ is generated as an algebra by the image of t and S. The elements $s^{-1}, s \in S$, are in the image of t. We put $s^{-1} = t(s^{-1})$. The algebra F_* is generated by the image of λ' and the s^{-1}, $s \in S$, because if $s = e(V)$ then $\beta'(s^{-1})$ in $U_{2|V|-1}(BG)$ is represented by $SV/G \to BG$ and these elements generate $\tilde{U}_*(BG)$ as U_*-module. Moreover it is sufficient to take only s of the form D^k, $D = e(V)$, where V is a fixed irreducible G-module.

Given $x \in U_n{}^G$, we can write

$$\lambda x = \Sigma s_i t(x_i), \qquad s_i \in S, \qquad x_i = \Sigma x_{ij} D^{-j}.$$

Hence there is an integer $m \geqslant o$ such that $D^m \lambda x$ is contained in the algebra generated by S and image (λi). If $m > o$ put

$$D^m \lambda x = \lambda i y + \Sigma(\lambda i y_j) s_j, \qquad s_j \neq 1. \tag{2}$$

We have relations of the following type

$$s_j = D u_j, \tag{3}$$

where u_j is contained in the algebra generated by S and image (λi). It is sufficient to prove this for $s = s_j = e(V)$, V irreducible. Since $U_1(B\mathbf{Z}_p) = \mathbf{Z}_p$ there is an integer a such that $a\beta'(s^{-1}) = \beta'(D^{-1})$ and hence there is $z \in \mathscr{U}_2{}^G$ such that $D(a - (\lambda' z)s) = s$. If we combine (2) and (3) we get

$$D^{m-1}\lambda x = D^{-1}\lambda i y + \Sigma(\lambda i y_j) u_j.$$

We apply β and get

$$o = \beta(D^{m-1}\lambda x) = \beta(D^{-1}\lambda i y) = \beta'(D^{-1}\lambda' y).$$

Hence there is $y' \in \mathscr{U}_*{}^G$ such that $\lambda i y' = D^{-1}\lambda i y$. The relation

$$D^{m-1}\lambda x = \lambda(i y' + \Sigma(i y_j) u_j)$$

gives that $D^{m-1}\lambda x$ is contained in the algebra generated by S and image (λi) and hence by induction also λx. The assertion (c) follows easily.

§6. CHARACTERISTIC NUMBERS

We assume $G = \mathbf{Z}_p$. The map α can be computed from the localized map $S^{-1}\alpha$. It is not difficult to see, that α is injective if $S^{-1}\alpha$ is injective.

PROPOSITION 6.1. *The map $S^{-1}\alpha$ is injective for $G = \mathbf{Z}_2$.*

Proof. This is an easy consequence of results of Conner [9]. We compare our map $S^{-1}\alpha$ with the map ∂' of [9], p. 87. The range of ∂' coincides with the integral part in degree zero of $S^{-1}U^*(B\mathbf{Z}_2)$, and ∂' is essentially the map $S^{-1}\alpha \circ (U_*(n) \otimes \mathrm{id})$ (see Proposition 4.1). The result follows from [9, Theorem 14.1].

We come now to characteristic numbers. Let $K^*(X)$ be $\mathbf{Z}$-graded complex K-theory and let $\mathbf{Z}[a_1, a_2, \ldots]$ be a polynomial ring in indeterminates $a_1, a_2, \ldots$ (of degree zero). There exists a unique multiplicative stable natural transformation of degree zero

$$B: U^*(X) \to K^*(X) \hat{\otimes} \mathbf{Z}[a_1, a_2, \ldots],$$

such that the Euler class of the line bundle η is mapped to

$$(\eta - 1) + (\eta - 1)^2 \otimes a_1 + (\eta - 1)^3 \otimes a_2 + \cdots.$$

If X is a point then B is an embedding as a direct summand (Hattori [19], Stong [22]). B defines a natural transformation of cohomology theories and hence a transformation of the corresponding spectral sequences. On the E_2-level this transformation is an embedding as a direct summand. If the K-theory spectral sequence is trivial (e.g. $X = BG$), then also the U^*-theory spectral sequence and B induces on the E_∞-level an injective map. Hence B itself is injective.

If we expand Bx with respect to the basis of $\mathbf{Z}[a_1, a_2, \ldots]$ consisting of monomials in the $a_1, a_2, \ldots$ we consider the resulting coefficients as K-theory characteristic numbers. Combining Proposition 6.1, Theorem 5.1.(c) and the remarks above we see that the map $B\alpha$ is injective for $G = \mathbf{Z}_2$. We express this fact in the next proposition.

PROPOSITION 6.2. *The bordism class of a unitary $\mathbf{Z}_2$-manifold is determined by its K-theory characteristic numbers.*

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